Exploring blurred choice axioms for constructive reverse mathematics

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Abstract

We report on work in progress on generalising the blurring technique due to Kirst and Zeng to the full axiom of choice. They use blurring applied to the drinker paradox and variants of dependent choice to carry out a constructive reverse analysis of the downward Löwenheim-Skolem theorem (DLS) for countable signatures. Our work aims at generalising their work and that of Espindola and Karagila in classical settings to DLS for arbitrary signatures through blurred axioms of choice. We verify our results using Rocq.

The downward Löwenheim-Skolem theorem (DLS) states that any model over infinite signature Σ can be turned into a model with cardinality of Σ [12]. It was already remarked by Skolem that using a variant of the axiom of choice seems to be necessary for DLS, and it is an exercise in the book by Boolos and Jeffrey [1] that, indeed, DLS for countable signatures is *equivalent*, over ZF, to the axiom of dependent choice.

The discipline of proving reverse implications to show necessity is called *reverse mathematics* [4, 11]. In *constructive reverse mathematics* [7] one furthermore works over a constructive foundation without the law of excluded middle. Kirst and Zeng [10] discover that the mentioned reverse result for DLS has much more hidden structure: DLS for countable signatures is equivalent to some (weaker) consequences of dependent choice and some (weaker) consequences of the law of excluded middle. To do so, they introduce a technique called *blurring*. For example, the blurred drinker paradox states:

$$\mathrm{BDP}^B := \forall X.\, X \to \forall P : X \to \texttt{Prop}.\, \exists f : B \to X.\, (\forall b.\, P(fb)) \to \forall y.\, Py$$

where BDP with B the unit type is exactly the usual drinker paradox [13] and in fact equivalent to LEM [10]. Kirst and Zeng also introduce blurred countable choice (BCC) and directed dependent choice (DDC), both provable from dependent choice. It results in:[1]

$$\mathrm{DLS} \leftrightarrow \mathrm{BDP}^{\mathbb{N}} \wedge \mathrm{BCC} \wedge \mathrm{DDC}$$

While BCC is the blurred counterpart of CC, DDC appears as a by-product in the decomposition of DC. It states that from a directed relation, one can extract a countable directed sub-relation.

The status of DLS for arbitrary signatures has been studied by Espíndola and then by Karagila. Espíndola proves that DLS for arbitrary infinite signatures is equivalent to the full axiom of choice [3], whereas Karagila proves more local results relating DLS for infinite signatures of fixed cardinality to the axiom of choice for fixed cardinality [8]. In this work, we aim at understanding blurring for the full axiom of choice, especially but not only in order to analyse the status of DLS for arbitrary signatures in constructive reverse mathematics, generalising at the same time the results of Kirst and Zeng as well as Espíndola and Karagila.

We work in Rocq's type theory with a syntactic universe of propositions `Prop`, without any extensionality assumptions, proof irrelevance, or unique choice. All stated results are verified in Rocq using the Library for First-Order Logic [9].

Classical and blurred choice axioms The classical axiom of choice states that every total relation has a choice function:

$$\mathrm{AC}_{X,Y} := \forall R : X \to Y \to \texttt{Prop}.\ \mathrm{total}(R) \to \exists f : X \to Y.\, \forall x.\, Rx(fx)$$

We use the shorthand $\mathrm{AC}_X := \forall Y.\ \mathrm{AC}_{X,Y}$ and similarly treat omitted indices as universally quantified. AC is a rather strong statement, stronger than the law of excluded middle in

[1]We in this abstract only consider results for first-order logic without existential quantification, therefore avoiding an additional principle dual to BDP called BEP in [10].

extensional settings, as per the Goodman-Myhill-Diaconescu theorem [5, 2]. Note that we work in a non-extensional setting, where this implication does not necessarily hold.

Kirst and Zeng [10] introduce the blurred axiom of countable choice (BCC) and end their paper with a generalisation to a blurring for the full axiom of choice (BAC). Recall:

$$\mathrm{BCC}_Y := \forall R : \mathbb{N} \to Y \to \texttt{Prop}.\ \mathrm{total}(R) \to \exists f : \mathbb{N} \to Y.\, \forall n : \mathbb{N}.\, \exists m : \mathbb{N}.\, Rn(fm)$$

$$\mathrm{BAC}_{X,Y} := \forall R : X \to Y \to \texttt{Prop}.\ \mathrm{total}(R) \to \exists f : X \to Y.\, \forall x : X.\, \exists x' : X.\, Rx(fx')$$

However, BAC does not allow fixing a blurring type B. We introduce a more controlled version:

$$\mathrm{CBAC}^B_{X,Y} := \forall R : X \to Y \to \texttt{Prop}.\ \mathrm{total}(R) \to \exists f : B \to X \to Y.\, \forall x : X.\, \exists b : B.\, Rx(fbx)$$

Now in order to summarise a few properties, we first record how CBAC generalises BAC:

Fact 1. $\mathrm{BCC} \leftrightarrow \mathrm{CBAC}^{\mathbb{N}}_{\mathbb{N}}$ *and more generally* $\mathrm{BAC}_X \leftrightarrow \mathrm{CBAC}^X_X$ *whenever there exists a surjective function* $z : X \to X \times X$.

Proof. For the more general equivalence, whenever given a total relation $R : X \to Y \to \texttt{Prop}$ one can turn a blurred choice function $f : X \to Y$ in the sense of BAC_X into $f' : X \to X \to Y$ being one in the sense of CBAC^X_X by simply setting $f'xx' := fx$. Conversely, if $g : X \to X \to Y$ is a blurred choice function in the sense of CBAC^X_X, then $g'x := g(zx)$ is one in the sense of BAC_X. □

Note that the left-to-right direction of the general equivalence does not require z. We record three facts about CBAC that share similarities with other blurred axioms studied in [10]:

Fact 2.
1. $\mathrm{CBAC}^{\mathbb{1}}_{X,Y} \leftrightarrow \mathrm{AC}_{X,Y}$.
2. *If there is a surjection of type* $B' \to B$, *then* $\mathrm{CBAC}^B_{X,Y} \to \mathrm{CBAC}^{B'}_{X,Y}$.
3. *If* B *is inhabited, then* $\mathrm{AC}_X \leftrightarrow \mathrm{CBAC}^B_X \wedge \mathrm{AC}_{X,B}$ *holds.*

Example usage in constructive reverse mathematics We now illustrate the expected role of BAC in generalising the results of Kirst and Zeng to cover the extensions of Espíndola and Karagila. First, for a suitable statement of DLS, given a signature Σ we let DLS_Σ state that for every inhabited model M of Σ there is a model N elementarily embedded into M whose domain is the type $\mathbb{F}$ of formulas over Σ. We then obtain the following lower bound for the strength of DLS_Σ regarding CBAC:

Fact 3. *Let* Σ *only contain unary predicate symbols. Then,* DLS_Σ *implies* $\mathrm{CBAC}^{\mathbb{F}}_\Sigma$.

Proof. Let $R : \Sigma \to Y \to \texttt{Prop}$ be total. We turn Y into a model for Σ by setting $p^Y := Rp$ for every $p : \Sigma$. Then, totality of R implies that Y satisfies all formulas $\exists y.\, p(y)$. Now, DLS_Σ provides a model N over domain $\mathbb{F}$ together with an elementary embedding $h : \mathbb{F} \to Y$. Defining $h' : \mathbb{F} \to \Sigma \to Y$ by $h'\varphi p := h\varphi$ yields a blurred choice function in the sense of $\mathrm{CBAC}^{\mathbb{F}}_\Sigma$. □

Note that the fact that in this proof, the blurred choice function does not rely on Σ hints at a stronger variant of CBAC that remains to be explored. Additionally, it is ongoing research to formulate a variant DDC^* of DDC that makes this characterisation tight, i.e. that establishes the following equivalence:

Conjecture 1. $\mathrm{DLS}_\Sigma \leftrightarrow \mathrm{BDP}^{\mathbb{F}} \wedge \mathrm{CBAC}^{\mathbb{F}}_\Sigma \wedge \mathrm{DDC}^*$

As a preliminary result, we proved that in fact a stronger instance $\mathrm{CBAC}^{\mathbb{F}}$ is enough to derive another variant of DLS_Σ, then even without any need for DDC, thus providing an upper bound [6]. Also, a variation of DLS_Σ implying $\mathrm{BDP}^{\mathbb{F}}$ has already been proved in [10].

Future work We identify two main possible continuation of this work, in addition to proving that the conjecture does hold. We first want to work on other constructive reverse analyses in model theory, such as for the upward Löwenheim-Skolem theorem, the completeness theorem . . . and find out whether the blurred logical principles apply on these theorems. We also want to find a clear definition of blurring and understand its properties. Both these paths would contribute to better understanding of the link between constructive and classical reverse mathematics.

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