Exploring blurred choice axioms for constructive reverse mathematics

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Classical reverse status of the Downward Löwenheim-Skolem theorem

Theorem (Downward Löwenheim-Skolem)
Every model of first-order logic with signature of cardinality κ
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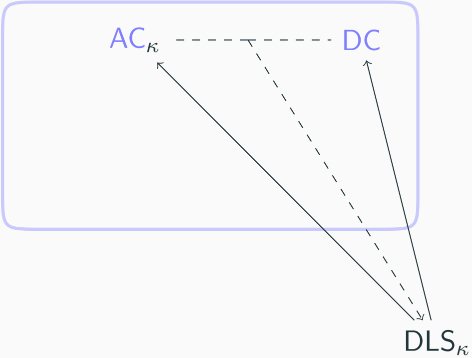
Karagila (2014), strengthening of results by Espindola (2012)

Theorem 4. *For every* $\aleph$ *cardinal* κ, $\mathsf{LS}(\kappa)$ *is equivalent to the conjunction of* DC *and* AC_κ.

Classical reverse status of the Downward Löwenheim-Skolem theorem



Choice axioms

Karagila (2014)

Constructive reverse status of the Downward Löwenheim-Skolem theorem

AC_κ DC

LEM

DLS_κ

Diaconescu's theorem not applying (lack propositional extensionality)

Choice axioms

Classical axioms

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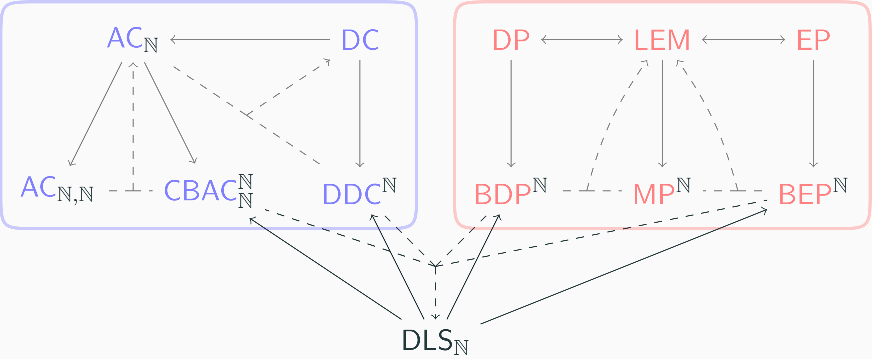

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Kirst and Zeng (2026)

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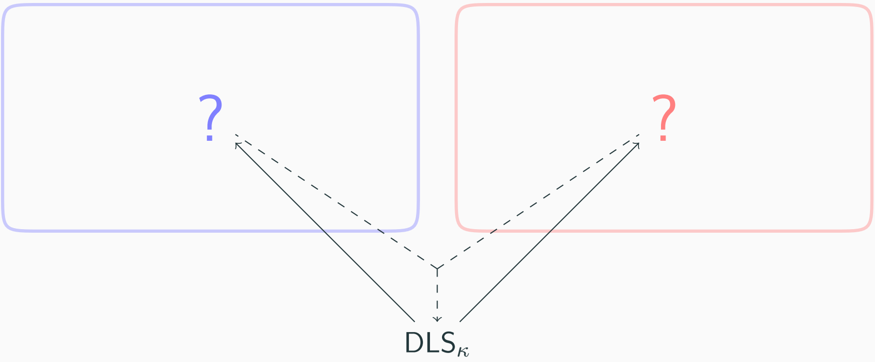

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Choice axioms

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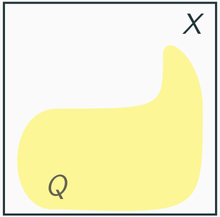
Blurred Drinker Paradox

Drinker Paradox

$\mathsf{DP}_X := \forall Q : X \to \mathbb{P}.\, \exists w : X.\, Qw \to \forall x : X.\, Qx$



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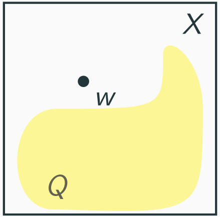

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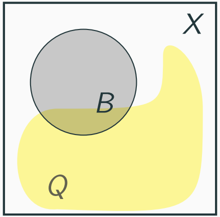

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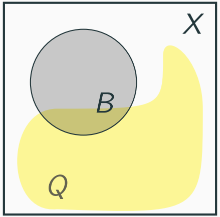

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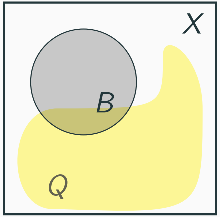

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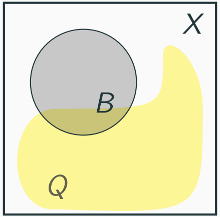

Decomposition of LEM

LEM

- LEM is the Law of Excluded Middle.

Kirst and Zeng (2026)

Decomposition of LEM

DP $\longleftrightarrow$ LEM

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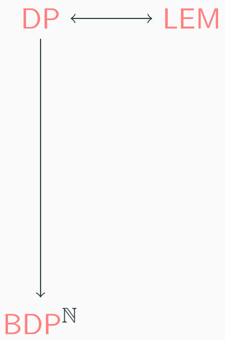
Decomposition of LEM



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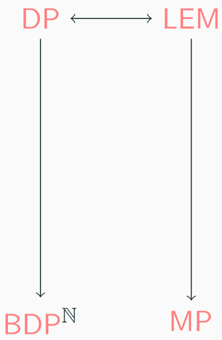
Decomposition of LEM



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- MP is Markov's Principle .

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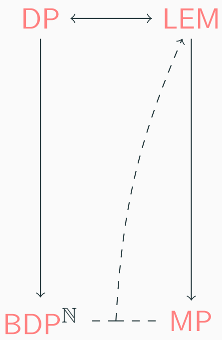

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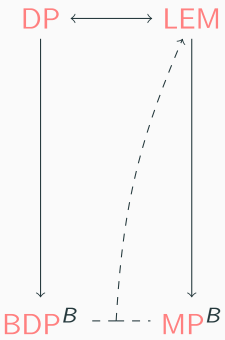

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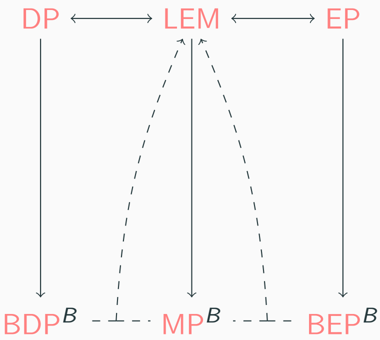

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Blurred Axiom of Choice

Axiom of Choice

$$\mathsf{AC}_{X,Y} := \forall R : X \to Y \to \mathbb{P}.\ \mathsf{total}\, R \to \exists f : X \to Y.\, \forall x.\, Rx(fx)$$

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Controlled Blurred Axiom of Choice

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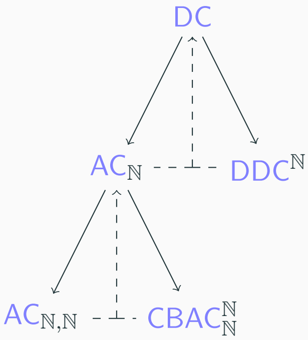
Fact

If there exists a surjection $X \twoheadrightarrow X \times X$, then the following are equivalent:

- $\mathrm{CBAC}^{X}_{X,Y}$*;*
- $\mathrm{BAC}_{X,Y}$ *as in Kirst and Zeng (2026) (related to the axiom of collection in Castro's work (2024)).*

Decomposition of AC and DC



Kirst and Zeng (2026)

- DC is Dependent Choice.
- $AC_{\mathbb{N}}$ is the Axiom of Choice on domain $\mathbb{N}$, often called Countable Choice.
- $AC_{\mathbb{N},\mathbb{N}}$ is the Axiom of Choice on domain $\mathbb{N}$ and range $\mathbb{N}$.
- $CBAC^{\mathbb{N}}_{\mathbb{N}}$ is the Controlled Blurred Axiom of Choice on domain $\mathbb{N}$, with blur $\mathbb{N}$.
- $DDC^{\mathbb{N}}$ is Directed Dependent Choice with blur $\mathbb{N}$.

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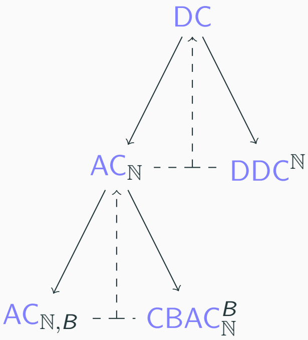

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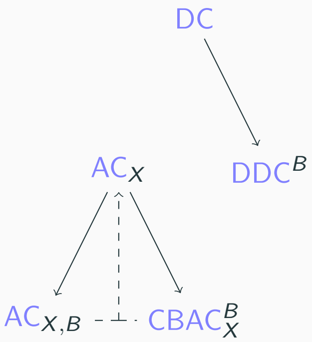

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- DDC^B is Directed Dependent Choice with blur B.

Downward Löwenheim-Skolem theorem

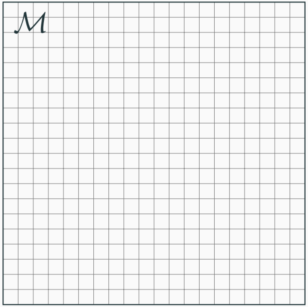
Definition (Elementary submodel)

A submodel $\mathcal{A}$ of a model $\mathcal{M}$ is elementary if for any formula $\varphi : \mathbb{F}$ and $v : A^{|\varphi|}$ parameters for φ, we have:

$$\mathcal{A} \models_v \varphi \leftrightarrow \mathcal{M} \models_v \varphi$$

Theorem (Downward Löwenheim-Skolem)

For any signature, assuming LEM and AC, any model $\mathcal{M}$ admits an elementary submodel of domain $\mathbb{F}$.



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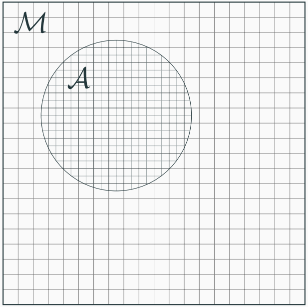

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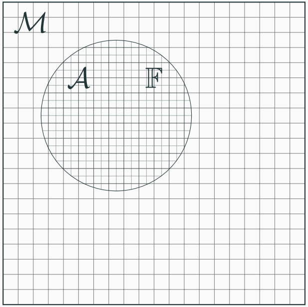

Constructive reverse status of DLS

Reverse question

What about the reverse status of DLS_Σ?

Constructive reverse status of DLS

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Theorem (Kirst and Zeng (2026))

$$\mathsf{CBAC}^{\mathbb{N}}_{\mathbb{N}} \land \mathsf{DDC}^{\mathbb{N}} \land \mathsf{BDP}^{\mathbb{N}} \land \mathsf{BEP}^{\mathbb{N}} \leftrightarrow \mathsf{DLS}_{\mathbb{N}}$$

Theorem (this work)

If $\mathbb{F}$ retracts into Σ:

$$\mathsf{CBAC}^{\mathbb{F}}_{\Sigma} \land \mathsf{DDC}^{\mathbb{F}} \land \mathsf{BDP}^{\mathbb{F}} \land \mathsf{BEP}^{\mathbb{F}} \leftrightarrow \mathsf{DLS}_{\Sigma}$$

Forward

Sketch

1. define a relation R on subsets of $\mathcal{M}$;
2. show that any directed sub-relation of R involving K many subsets yields an elementary submodel retracting into K;
3. show that this R is directed using $\mathsf{CBAC}_{\mathbb{F}}^{\mathbb{F}}$ and $\mathsf{BDP}^{\mathbb{F}}$ and $\mathsf{BEP}^{\mathbb{F}}$;
4. find a directed sub-relation of R involving $\mathbb{F}$ many subsets using $\mathsf{DDC}^{\mathbb{F}}$;
5. get an elementary submodel of $\mathcal{M}$ retracting into $\mathbb{F}$.

Backward: $\mathsf{DLS}_\Sigma \to \mathsf{BDP}^{\mathbb{F}}$

DLS_Σ

$\forall \mathcal{M}.\, \exists \mathcal{A} \subseteq \mathcal{M}.\, |\mathcal{A}| = \mathbb{F} \land \forall (\varphi : \mathbb{F})(\forall v : A^{|\varphi|}).\, \mathcal{A} \models_v \varphi \leftrightarrow \mathcal{M} \models_v \varphi$

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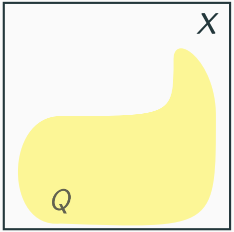

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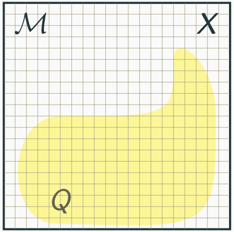
Assume unary $p : \Sigma$. Define $\mathcal{M}$ with domain X and



$$p^{\mathcal{M}} x := Qx$$

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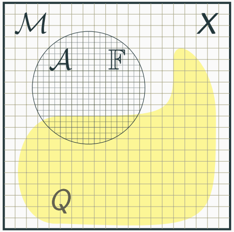

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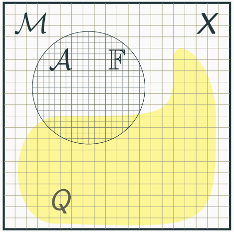

$$p^{\mathcal{M}} x := Qx$$

Obtain $\mathcal{A} \subseteq \mathcal{M}$ via w.

1. Assume $\forall b : \mathbb{F}. Q(wb)$, we need to prove $\forall x. Qx$
2. By elementarity on $\forall x. px$, we need to prove $\forall b : \mathbb{F}. p^{\mathcal{A}} b$.
3. Fix $b : \mathbb{F}$. By elementarity on pb, we need to prove $Q(wb)$.
4. Which follows by assumption.

What's next?

Next steps

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Thank you!

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